Journal of Sound and Vibration (1999) **222**(1), 152–162
Article No. jsvi.1998.2014, available online at http://www.idealibrary.com on IDEAL

ON THE PROPAGATION OF SMALL PERTURBATIONS IN TWO SIMPLE AEROELASTIC SYSTEMS

A. Iollo and M. D. Salas
Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23681-2199, *U.S.A.*

(*Received* 25 *September* 1998, *and in final form* 19 *October* 1998)

1. INTRODUCTION

The motivations for this study are twofold. On the one hand we wanted to study the effects of the mean flow on the acoustic-wave speed, in the presence of a coupling with a structural element bounding the fluid. On the other hand, since the energy of a perturbation is partitioned between fluid and structure according to its speed, we were interested in investigating how the Mach number of the undisturbed flow may affect the noise scattered at inhomogeneities by the structure.

In classical papers on aeroelastic interactions, the time evolution of small perturbations is studied [1]. The stability boundaries are determined as functions of a speed parameter (the ratio of the wave velocity in the panel in the absence of coupling and the wavelength of the disturbance). It is also found that a panel characterized elastically by flexural forces only is unstable at any finite airspeed for sufficiently large wavelengths, whereas the introduction of membrane tension will lead to instability only for airspeeds greater than the minimum wave velocity of the panel. More recently the same problem was studied from another viewpoint, the interest being the scattering of a bending wave by an inhomogeneity in an otherwise homogeneous and infinite panel immersed in a fluid at rest [2]. Given a certain frequency of the perturbation, the dispersion relation of the coupled system is studied in terms of the wave number, whereas in the study of stability, a frequency analysis was preferred in order to detect the eventual time-wise growth of the propagating wave.

The study of the dispersion relation for a homogeneous beam is a preliminary step in analyzing the behavior of the air–beam system in the presence of inhomogeneities. In fact, the effect of gaps, stiffeners etc., is accounted for by the presence, in the right-hand side of the beam equation, of a linear combination of the Dirac function and its derivatives. The right-hand side of the beam equation amounts to a forcing on the system whose response is, in the Fourier space, the ratio between the Fourier transform of the forcing term and the Fourier transform of the dispersion relation. Therefore, in the physical space, the

0022–460X/99/160152 + 11 $30.00/0

solution is governed by the poles of such ratio, which are in turn the zeros of the dispersion relation.

In what follows, a simple one-dimensional configuration was first studied in which the Mach number plays a role only on the stability bounds, while in the two-dimensional case the Mach number has an important effect on the solution of the dispersion relation, allowing or not certain waves to appear. The one-dimensional case, however, has the merit of showing clearly the influence of the fluid–structure coupling on the speed of the propagating waves. One may anticipate that for low values of the stiffness, the propagation speeds in the beam and in the fluid are remarkably different from those in the uncoupled case.

2. QUASI ONE-DIMENSIONAL COUPLING

The flow of a compressible fluid is studied through a nozzle with elastic walls. The nozzle walls are loaded by the pressure difference between an outside ambient pressure and the local internal fluid pressure. The flow is assumed to be quasi one-dimensional, inviscid and isentropic. Under these hypotheses the non-dimensional equations governing the flow are the following

$$\frac{2}{\gamma-1}c_t + cu_x + \frac{2}{\gamma-1}c_x u + \frac{c}{H}(H_t + uH_x) = 0, \tag{1}$$

$$u_t + uu_x + \frac{2}{\gamma-1}cc_x = 0, \tag{2}$$

where c is the local speed of sound, u is the velocity of the fluid, H the nozzle height and γ the specific heats ratio.

In addition, it is assumed that the deformation of the walls of the nozzle are so small that the motion is governed by the linear beam equation

$$mH_{tt} + DH_{xxxx} = p_i - p_0, \tag{3}$$

where D is the bending stiffness, p_i the local pressure of the fluid, p_0 is the outside ambient pressure and m the linear mass of the walls, which is 1 in what follows.

The coupling between the quasi one-dimensional fluid equation and the beam equation, which is due to the pressure difference on the right-hand side of equation (3), is interesting because of the different nature of the partial differential equations (PDEs) governing the fluid and the nozzle wall motion. If only the fluid is considered, one has a hyperbolic system of PDEs representing signals that propagate on two characteristics with speeds $u \pm c$. The perturbations are felt in the fluid only after a finite time, needed for the perturbation to propagate from the source to the receiver. On the other hand, the linear beam equation is parabolic, i.e., perturbations are immediately felt all along the beam, although the phenomena is still evolving in time. In fact, from the dispersion relation of this PDE one has two waves travelling with speeds $\pm\sqrt{D}k$ and two near fields [3].

The coupled system is parabolic, but the travelling waves of each uncoupled system play an important role for what concerns the stability of the solution and the partition of the energy of the perturbations between the fluid and the nozzle walls.

Let us consider a nozzle with straight walls at $t = 0$ and with an inlet Mach number M_0. We want to study the evolution of small perturbations for this system. Take $c = c_0 = c'$, $u = u_0 + u'$ and $H = H_0 + H'$ and substitute into equations (1), (2) and (3). Assuming that $p_0 = \rho_0 = 1$, that the prime quantities are small, and dropping the prime notation, one obtains the following system for the perturbations:

$$\frac{2}{\gamma-1}c_t + c_0 u_x + \frac{2}{\gamma-1}u_0 c_x + \frac{c_0}{H_0}(H_t + u_0 H_x) = 0, \tag{4}$$

$$u_t + u_0 u_x + \frac{2}{\gamma-1}c_0 c_x = 0, \tag{5}$$

$$H_{tt} + D\,H_{xxxx} - \frac{2}{\gamma-1}c_0\left(\frac{c_0^2}{\gamma}\right)^{\frac{1}{\gamma-1}} c = 0. \tag{6}$$

This system of PDEs governs the evolution of small disturbances in a nozzle with parallel elastic walls. Assuming that the solution has the form

$$c = \hat{c}\,\mathrm{e}^{i(kx-\omega t)}, \quad u = \hat{u}\,\mathrm{e}^{i(kx-\omega t)}, \quad H = \hat{H}\,\mathrm{e}^{i(kx-\omega t)}, \tag{7–9}$$

substitute into equations (4)–(6) to get

$$\hat{u} = \frac{2}{\gamma-1}\frac{c_0 k}{\omega - u_0 k}\hat{c}, \quad \hat{H} = \frac{2}{\gamma-1}c_0\left(\frac{c_0^2}{\gamma}\right)^{\frac{1}{\gamma-1}}\frac{\hat{c}}{Dk^4-\omega^2}, \tag{10, 11}$$

$$(\omega - u_0 k)^2\left[1 + \frac{c_0^2}{H_0(Dk^4-\omega^2)}\left(\frac{c_0^2}{\gamma}\right)^{\frac{1}{\gamma-1}}\right] - c_0^2 k^2 = 0. \tag{12}$$

Note that as $D \to \infty$ or $k \to \infty$ the system becomes increasingly uncoupled, i.e., the evolution of the perturbations in the beam are less and less influenced by the presence of the fluid and vice versa.

For a given wave number k, one may solve equation (12) with respect to ω. When $\mathrm{Im}(\omega) \neq 0$ the corresponding mode of oscillation is unstable. Figure 1 shows a plot of $\mathrm{Re}(\omega/k)$ with respect to D when $M_0 = 0$. The four solutions are obviously real and symmetric with respect to the abscissa. No unstable solution is possible since there is no forcing on the system.

The four solutions represent waves which travel in the positive and negative direction of the x-axis. They correspond to the waves present in each of the uncoupled systems which have speeds $\pm c_0$ and $\pm\sqrt{D}k$. Solutions b and c go to 0 when the stiffness is zero (no wall separating the ambient and internal flow), while the waves corresponding to solutions a and d have speeds equal to that of

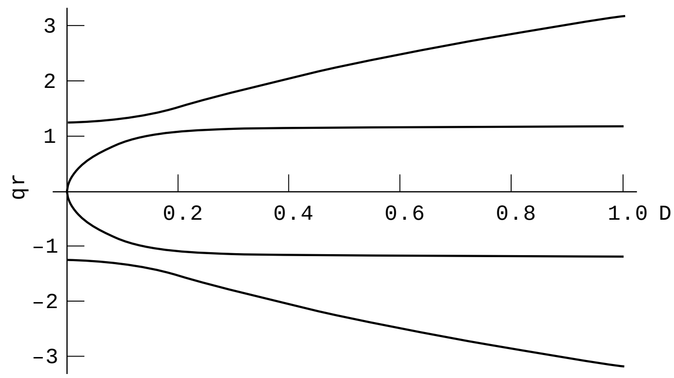

Figure 1. Solutions of the dispersion relation for a quasi one-dimensional coupling. $c_0 = \sqrt{1{\cdot}4}$, $M_0 = 0$, $k = \pi$ and $qr = \mathrm{Re}(\omega/k)$. The branches of the solution are named a, b, c, d from top to bottom.

the signals in the fluid $\pm c_0$. For increasing stiffness the solutions gradually shift role. For example, the solutions that are 0 for $D = 0$ asymptotically approach the value of $\pm c_0$ when the system becomes uncoupled, i.e., for $D \to \infty$. Conversely, a and d approach the curves $\omega/k = \pm\sqrt{D}k$ as $D \to \infty$.

The partition of the energy of the perturbations between beam and fluid depends on the phase speed ω/k, of the wave considered. It is seen from equation (11) that for a given amplitude $\hat{c}$, if the speed of the wave considered is close to $\pm\sqrt{D}k$ then $\hat{H} \to \infty$ This means that when the speed of a wave in the coupled system is close to the speed of a wave present, for example, in an isolated beam, the energy of the perturbation is mainly concentrated in the beam. Similar arguments can be made for waves whose energy is mainly in the fluid.

Figure 2 illustrates the case corresponding to $M_0 = 0{\cdot}5$. Now it is found that there is a range of values of D where $\mathrm{Im}(\omega/k) \neq 0$ for the solutions b and c. The existence of this region indicates that unstable motion can be triggered by small disturbances with a given wave number. Note that $\mathrm{Im}(\omega/k) \neq 0$ corresponds to the small region in Figure 2 where the branches b and c collapse into one curve, i.e., the speed of propagation of the two waves is the same. This is necessarily the case since the dispersion relation is a fourth order polynomial in ω. Note also that the solutions b and c have asymptotes c_0 $(0{\cdot}5 \pm 1{\cdot}0)$.

Because the unstable modes are associated with the collapsed branches b and c, it can be concluded that their energy is mostly in the beam. Interestingly, there is a range of values for the stiffness for which the unstable modes can propagate only in the positive direction.

2.1. COMPUTATIONAL EXPERIMENT

A simply supported beam of unit length is considered which is in contact with a fluid at rest governed by equations (1) and (2) on one side, and to a constant ambient pressure equal to that of the unperturbed fluid on the other side. A simply supported beam is used so that there are no near fields generated at the

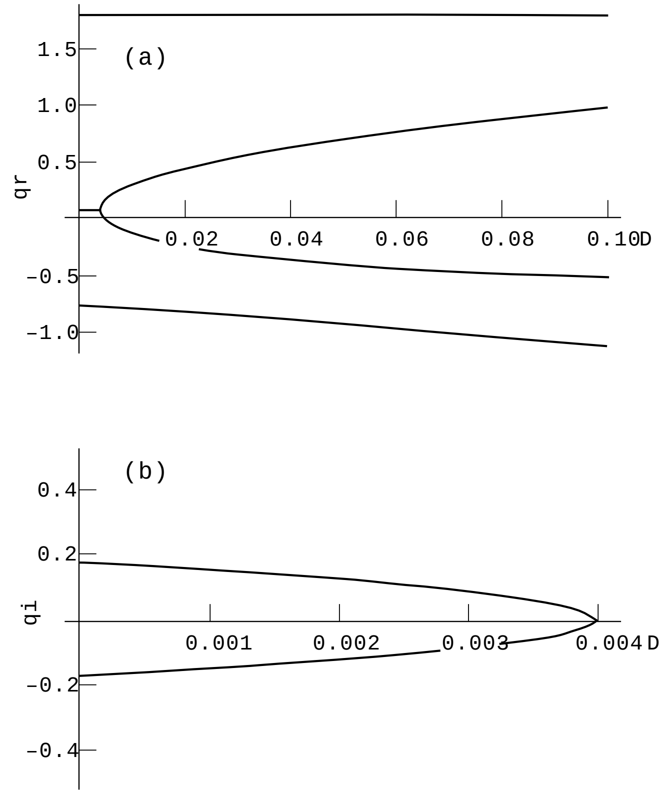

Figure 2. Solutions of the dispersion relation for a quasi one-dimensional coupling. $c_0 = \sqrt{1 \cdot 4}$, $M_0 = 0 \cdot 5$, $k = \pi$: (a), $qr = \mathrm{Re}(\omega/k)$, (b) $qr = \mathrm{Im}(\omega/k)$. The branches of the solution are named a, b, c, d from top to bottom of Figure 1.

boundaries [3]. The flow takes place between the elastic beam and a rigid wall. This elastic "hose" connects two reservoirs whose pressure is kept constant and equal to that of the unperturbed flow in the hose. Therefore, the boundary points are nodal points for the pressure and displacement waves as well. When the beam is displaced from its equilibrium position it will perform free periodic oscillations corresponding to a superposition of the modes excited by the initial condition. There is no dissipative external force acting on the system and the system is conservative.

The beam equation was solved by means of a semi-discretization based on a Galerkin projection of the solution on the eigenmodes of an isolated simply supported beam. This results in the solution of a set of ordinary differential equations (ODEs) for each mode taken into account. The ODEs are then integrated in time by means of a standard fourth order Runge–Kutta scheme. Besides providing high resolution, this approach allows one to control very closely the modes of the coupled system excited by the initial condition which drives the system out of equilibrium. The given initial condition is the beam displacement. In particular, the beam is displaced so that only the first mode of oscillation has non-null amplitude, i.e., $H(x, 0) = h \sin \pi x$ with small h. Thus, one is able to impose the wave number of the free oscillations in order to compare the frequencies resulting from the simulation with that computed by equation (12). Other modes of oscillation have amplitudes of much lower order compared to that excited.

The fluid equations are discretized by a finite-volume scheme where the fluxes at the volume interfaces are computed as in reference [4]. Higher order accuracy is achieved by means of an ENO algorithm; see reference [5]. The number of computational volumes used to discretize the flow equations is 1000, so that the accuracy of the results is of the order of 10^{-6}. The computations were run in double precision.

In Figure 3 is plotted the Mach number at the inlet of the nozzle versus time. It is seen that two frequencies of oscillation are present. Because of the set-up of the experiment, the perturbation is not travelling, but forming a standing wave in the nozzle (standing waves comprise travelling waves in both directions). The two frequencies of Figure 1 are the ones found in this experiment. In particular it was verified that the periods $T = 2\omega/\pi$ computed by equation (12) with $D = 0{\cdot}001$ (1·61, 21·1) are to a good approximation equal to those obtained with the numerical simulation (1·64, 21·7).

3. TWO-DIMENSIONAL COUPLING

Let us consider a two-dimensional case in which the equation governing the flow is the linear potential equation

$$(1 - M_0^2)\Phi_{xx} + \Phi_{yy} - \frac{1}{c_0^2}(2U_0\Phi_{xt} + \Phi_{tt}) = 0, \tag{13}$$

where $(\Phi_x, \Phi_y) = (u, v)$ are the components of the flow velocity vector, $(U_0, 0)$ and c_0 are, respectively, the velocity and speed of sound of the unperturbed flow, and $M_0 = U_0/c_0$.

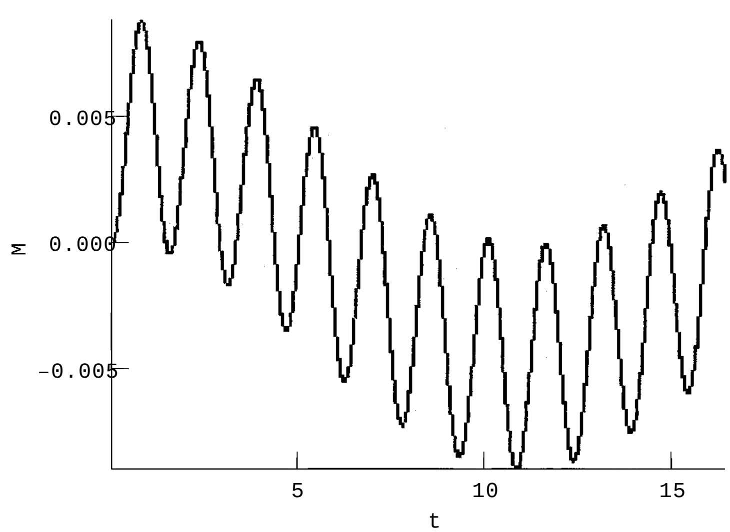

Figure 3. Mach number versus time at nozzle inlet: $D = 0{\cdot}001$ and $k = \pi$.

Consider an infinitely long flexible surface separating two regions of the flow. On this surface the boundary condition on the flow is given by the equation

$$\Phi_y = H_t + U_0 H_x, \tag{14}$$

where H is the distance of the flexible surface to the x-axis. In addition the potential Φ is required to vanish in the far field.

In the idealized system studied, it is assumed that the infinite surface is elastic and satisfies the linear small perturbation, beam equation

$$H_{tt} + DH_{xxxx} = [p] = 2\rho_0(\Phi_t + U_0\Phi_x), \tag{15}$$

where ρ_0 is the fluid density of the unperturbed flow and $[p]$ is the pressure jump across the wall. For simplicity, it is assumed that the flexible surface is wetted by the fluid on both sides. The case corresponding to a flow at rest on one side leads to more complex algebraic manipulations, but the conclusions would not be altered.

Equations (13)–(15) form a coupled system. The coupling comes about through the aerodynamic load on the moving surface (beam) and the boundary condition, equation (14).

Our study is limited to such a linear model since we are interested in studying how the coupling affects the propagation of small amplitude waves. To do that, take

$$\Phi = \hat{\Phi}\exp[i(k_1 x + k_2 y - \omega t)], \quad H = \hat{H}\exp[i(k_1 x - \omega t)], \tag{16, 17}$$

and substitute these expressions into equations (13)–(15). The angular velocity ω is supposed to be a real number. Therefore, waves whose amplitude are not diverging or decaying in time are considered.

Solving for k_2 in equation (13) one obtains

$$k_2 = \sqrt{\left(\frac{\omega}{c_0} - M_0 k_1\right)^2 - k_1^2}. \tag{18}$$

Note that to have a finite amplitude wave for large y and to ensure the radiation condition, i.e., outgoing waves in the far field, k_2 is either a positive imaginary or a positive real:

$$k_2 \in iR^+ \quad \text{or} \quad k_2 \in R^+. \tag{19}$$

These conditions are a very important discriminant for admitting or not certain solutions and use will be made of them later.

From equation (14) one has

$$\hat{\Phi} = -\frac{\omega - U_0 k_1}{k_2}\hat{H} \tag{20}$$

and from equation (15) one obtains

$$(Dk_1^4 - \omega^2)\hat{H} = -2\rho_0 i(\omega - U_0 k_1)\hat{\Phi}. \tag{21}$$

Substituting equations (18) and (20) into equation (15), and making use of equations (16) and (17) one obtains the dispersion relation for the coupled aeroelastic system.

The dispersion relation is non-dimensionalized with respect to $K_0=(\omega^2/D)^{1/4}$, which is the wave number of the small perturbations travelling in the beam when there is no coupling with the fluid. Introducing also $k_0=\omega/c_0$, $\mu=k_0/K_0$, $K=k_1/K_0$ and $\nu=2\rho_0/K_0$, the dispersion relation is written as

$$K^4-1=\frac{i\nu}{\sqrt{(\mu-M_0K)^2-K^2}}. \tag{22}$$

The parameter μ has a physical meaning similar to that of the Mach number: it is the ratio between the speed of the perturbations in the beam to that in air when there is no coupling.

This equation relates the wave numbers and the frequencies of the small amplitude waves which can propagate in the coupled aeroelastic system. In the case of $M_0=0$ the above equation reduces to

$$K^4-1=\frac{\nu}{\sqrt{K^2-\mu^2}}, \tag{23}$$

which is identical to the dispersion relation obtained in reference [3, equation 3.9], for a case with zero mean flow. Notice that

$$k_2/K_0=\sqrt{(\mu-M_0K)^2-K^2},$$

which is the denominator of the right-hand side of equation (22).

Let us consider now the uncoupled system, where the beam vibration is not affecting the perturbations in the fluid and vice versa. In this case, the non-dimensional dispersion relation is

$$K_b^4-1=0, \tag{24}$$

with solutions

$$K_b=\pm 1,\ \pm i. \tag{25}$$

The solutions $K_b=\pm 1$ correspond to wave motion in the positive and negative directions of the x-axis. The solutions $K_b=\pm i$ represent near fields generated close to some boundary, these are used to accommodate the boundary conditions if present.

In the fluid, the acoustic waves propagating in the x direction have speed $\omega_f/k_f=U_0\pm c_0$, from which one can compute the dimensionless wave number

$$K_f=\frac{\mu}{M_0\pm 1}. \tag{26}$$

If it is assumed that the solutions of the coupled system are not very far from those of the uncoupled system equations (25) and (26), one can make equation (22) approximately solvable in closed form. Consider first the roots

$K \approx \pm 1$ and the case $|\mu \pm M_0| < 1$; after substituting into the right-hand side of equation (22) one has

$$K^4 - 1 = \frac{\nu}{\sqrt{1 - (\mu \mp M_0)^2}}, \tag{27}$$

where the conditions (19) were taken into account. To the same order of approximation the solution of the above equation can be written

$$K = 1 + \frac{\nu}{4\sqrt{1 - (\mu \mp M_0)^2}}, \tag{28}$$

Similarly for $|\mu \pm M_0| > 1$, one has

$$K = 1 + \frac{i\nu}{4\sqrt{(\mu \mp M_0)^2 - 1}} \tag{29}$$

These waves are equivalent to the waves that in an isolated beam travel from $-\infty$ to $+\infty$ without attenuation. In the coupled case, depending on $\mu \pm M_0$ one has two different behaviors. For $|\mu \pm M_0| < 1$, the wave number in the direction of the x-axis is real, while $k_2/K_0 \in iR^+$, i.e., the wave is decaying in the direction of the y-axis, and therefore, since there is no energy radiated away, it propagates without attenuation in the direction of the x-axis.

When $|\mu \pm M_0| > 1$, K has a non-zero imaginary part. The wave number in the direction of the y-axis is real, i.e., energy is radiated away from the vibrating beam and therefore the wave is decaying as it propagates along the beam.

The equivalent of the near fields existing in the uncoupled beam are found when $K \approx \pm i$:

$$K = \pm i\left(1 + \frac{i\nu}{4\sqrt{(\mu \mp M_0 i)^2 + 1}}\right), \tag{30}$$

which is valid for any value of $\mu \pm M_0$, therefore the type of solution found for the coupled aeroelastic system is basically the same as for the near fields corresponding to $M_0 = 0$.

The solutions corresponding to the acoustic waves are found rewriting equation (22) as

$$\sqrt{(\mu - M_0 K)^2 - K^2} = \frac{i\nu}{K^4 - 1}, \tag{31}$$

then assuming $K \approx \mu/(M_0 \pm 1)$, one has

$$\sqrt{\left(\mu - M_0 \frac{\mu}{M_0 \pm 1}\right)^2 - K^2} = \frac{i\nu}{(\mu/M_0 \pm 1)^4 - 1}. \tag{32}$$

The above equation has acceptable solutions, in the sense of the conditions (19), if and only if $|\mu/(M_0 \pm 1)| > 1$, which is equivalent to $|\mu \pm M_0| > 1$. In this case the solutions are

$$K = \frac{\mu}{M_0 \pm 1} + \frac{\nu}{2\left[\left(\frac{\mu}{M_0 \pm 1}\right)^4 - 1\right]} \frac{M_0 \pm 1}{\mu[(1 - M_0^2) + M_0\mu]}, \tag{33}$$

which are real numbers and therefore the waves travel without attenuation. The correspondent wave number in the direction of the y-axis is purely imaginary, so there is no radiation of energy to infinity. These waves are the equivalent to the acoustic waves in the fluid for the uncoupled system.

It should be noted that there are as many different kinds of waves as there are different systems interacting, and that the particular wave with its velocity near that of one of the component systems will entrust its energy chiefly to that component. This can be seen by substituting the solutions of the dispersion relations into equation (20), or equation (21), and solving for the ratio of the amplitudes.

Compared to the case in which $M_0 = 0$, there is a richer variety of solutions available, according to the inequality satisfied by $\mu \pm M_0$. In fact depending on the direction considered, one may have either $\mu + M_0 > 1$ or $-\mu + M_0 < 1$. In this case for example, the last pair of solutions obtained would propagate only in the positive direction of the x-axis.

This result is reasonable if one considers that what is important is the relative motion of the fluid with respect to the waves travelling in the beam, in this sense, it is interesting to compare the results in reference [3] where a similar analysis was done for $M_0 = 0$. In this case it is known that waves propagating in the beam in the x direction radiate energy in the y direction only if the wave is supersonic, i.e., $|\mu| > 1$. When $M_0 \neq 0$, a frame of reference at rest is taken with respect to the fluid. In the relative motion, the speed of the wave in the beam is $\mu \pm M_0$.

Why are these results relevant to the noise emission from a rib stiffener? Intuitively it is clear that when the wave energy is mostly into the fluid, very little energy is scattered at the stiffener, while if the wave energy is mostly concentrated in the beam, the noise emission will be higher. This argument can be made rigorous if one considers that the eigenvalues of the free aeroelastic system become the poles of the transfer function for the forced system constituted by the fluid, the beam and the stiffener. The number and the position of these poles in the complex plane are now functions not only of μ but of M_0 as well. Therefore, the emission of noise as a function of μ, as for example presented in reference [2], now depends on the free stream Mach number.

REFERENCES

1. J. W. MILES 1956 *Journal of the Aeronautical Sciences* **23**, 771–780. On the aerodynamic instability of thin plates.
2. M. S. HOWE 1986 *IMA Journal of Applied Mathematics* **36**, 247–262. Attenuation and diffraction of bending waves at gaps in fluid loaded plates.

3. L. Cremer, M. Heckl and E. E. Ungar 1973 *Structure-Borne Sound*. Berlin: Springer.
4. M. Pandolfi 1984 *AIAA Journal* **22**, 602–610. Contribution to the numerical prediction of unsteady flows.
5. A. Harten, B. Engquist, S. Osher and S. R. Chakravarthy 1987 *Journal of Computational Physics* **71**, 231–303. Uniformly high order accurate essentially non-oscillatory schemes III.